Optimal Turnover Constraints: *Scarcity Is Everywhere*

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A practical question facing global portfolio managers is how to optimally allocate a turnover budget to different markets. In practice, many regional alpha models are developed independently and then "stitched" together under the global optimization context. In the static or point-in-time perspective, there is no problem with this approach. When we take a dynamic viewpoint, however, the heterogeneity in alpha signals across different regions may cause issues. One example relates to the difference in "speed" of alpha signals across regions. Regions with faster-moving alpha signals may absorb more turnover. If we have a global turnover budget outlined in the investment mandate, then regions with slow-moving alphas would be "crowded" out in this turnover competition, even though they might have better alpha signals in terms of information ratio.

In response, the global portfolio manager may impose turnover constraints at the region or country level to force a more even distribution of turnover across regions. One key question remains, however: How should the portfolio managers determine the turnover allocation to each region? What characteristics of the regional alpha models should be accounted for? We attempt to address these questions in this article.

We take the dynamic model introduced by Grinold [2006, 2007] as our research platform and extend it to the multi-market scenario. In the single-market model, the optimal turnover depends on the transaction cost, speed of information flow, and risk-aversion parameter in a rather straightforward way. Once we move to the multi-market model, as will be discussed later in the article, the solution becomes more complicated. In particular, we find that a higher fraction of the turnover budget should be allocated to markets in which 1) the alpha model produces higher information ratios, 2) new information flows in more rapidly, or 3) it is less expensive to trade. However, the optimal turnover is quite insensitive to the aversion to risk from a given market.

Although this question is of notable practical importance, current literature remains surprisingly silent on it. One possible explanation is that turnover is by nature a dynamic concept, and dynamic portfolio optimization has only begun to flourish recently. From this perspective, our article is also related to Engle and Ferstenberg [2007], Qian, Sorenson, and Hua [2007], and Sneddon [2008]. By fine-tuning the weights of multiple alpha factors, Qian, Sorenson, and Hua [2007] proposed reducing the portfolio turnover by increasing the autocorrelation of information coefficients (ICs). Their objective, however, differed significantly from ours.

The organization of this article is as follows. First, we will review the single-market dynamic model introduced by Grinold [2007] and derive the portfolio turnover implied by the model. Then we will extend the model to the multi-market case with a total turnover constraint and present an implied function of the optimal turnover. Finally, we will illustrate the results with a numerical example and conclude.

MODEL

This section reviews the single-market model of Grinold [2007] and extends it to a multi-market case in order to derive an implied function of the optimal turnover.

Single-Market Model

The setup of the model is borrowed from Grinold [2007]. For the convenience of exposition, we will review some of the specification details.

The portfolio manager rebalances the portfolio at a constant interval of Δt years. For example, in the case of monthly rebalancing, Δt is 1/12. There are two portfolios—the model portfolio $m(t)$ and the holdings portfolio $p(t)$. The former is the optimal portfolio we would like to hold at time t, absent any transaction costs. The latter is the portfolio we actually hold after optimization with transaction costs.

The evolution of the model portfolio is assumed to be purely driven by the arrival of new information and the erosion of old information, which is characterized by the following equation:

$$\Delta m(t) = -g \cdot m(t - \Delta t) \cdot \Delta t + u(t) \tag{1}$$

where

- $\Delta m(t) = m(t) - m(t - \Delta t)$ is the change in the model portfolio at time t;
- g is the annual rate of information loss; and
- $u(t)$ is the new information arriving at time t, which is assumed to be independent of $m(t - \Delta t)$.

The actual portfolio $p(t)$ tracks the model portfolio $m(t)$ through trades according to the following equation:

$$\Delta p(t) = d \cdot \{m(t) - p(t - \Delta t)\} \cdot \Delta t \tag{2}$$

where d is the speed of trading, and $d \cdot \Delta t$ is the fraction of the gap between the model and the pre-trade portfolios closed at time t. If there were no transaction costs at all, we would like to trade to the exact model portfolio, and $d \cdot \Delta t$ would be equal to 100%. Although there are different interpretations of the variable d (for example, as an amortization factor for transaction cost, as pointed out by Grinold [2007]), we will see later that it is directly related to the level of portfolio turnover. Grinold has shown that the optimal value of d depends on the transaction cost level, which is consistent with the fact that portfolio managers often use transaction cost models to tune portfolio turnover.

If we define the risk of the holdings portfolio as

$$\omega_p = \sqrt{(E[p'(t) \cdot \Omega \cdot p(t)])}$$

of the model portfolio as

$$\omega_M = \sqrt{(E[m'(t) \cdot \Omega \cdot m(t)])}$$

and of the trades as

$$\omega_{\dot{p}} = \sqrt{\left(E\left[\left\{\frac{p(t) - p(t-\Delta t)}{\Delta t}\right\}' \cdot \Omega \cdot \left\{\frac{p(t) - p(t-\Delta t)}{\Delta t}\right\}\right]\right)}$$

where Ω is the asset variance-covariance matrix, then Equations (1) and (2) imply the following identities:[1]

$$\omega_p^2 = \frac{d}{d+g} \cdot \omega_M^2 \quad \text{and}$$

$$\omega_{\dot{p}} = gd \cdot \omega_p^2 = \frac{gd^2}{d+g} \cdot \omega_M^2 \tag{3}$$

Similarly, the transfer coefficient, which is the correlation of $m(t)$ and $p(t)$, can be derived as $\tau_p = \sqrt{\frac{d}{d+g}}$, and the information ratios of the model (IR_M) and of the portfolio (IR_p) have the following relationship:

$$IR_p = \sqrt{\frac{d}{d+g}} \cdot IR_M \tag{4}$$

Portfolio Turnover

Because our goal is to allocate the turnover budget in the most efficient way, we next want to infer the expected turnover level based on our model. Intuitively, we can posit that the turnover will depend on two factors—the speed of information flow, g, and the speed of trading, d. The speed of information flow determines how fast the alpha model updates itself; the higher the speed, the more turnover is needed. The speed of trading determines how closely the portfolio tracks the model; the narrower the gap, the greater turnover is required.

We can also derive the turnover from Equations (1) and (2). The derivation is not exactly rigorous, but can still provide us with some insights into the problem. Suppose at time $t - \Delta t$, we have a portfolio perfectly matched to the model, such as $p(t - \Delta t) = m(t - \Delta t)$. The question is, what will be the expected portfolio turnover at time t, or $E[\text{Turnover}(t) \mid p(t - \Delta t)]$?

The portfolio turnover can be defined as (we henceforth drop the $p(t - \Delta t)$ part of the notation)[2]

$$E[\text{Turnover}(t)] \equiv |E[\iota' \cdot \Delta p(t)/\iota' \cdot p(t - \Delta t)]|$$

which, after substituting Equations (1) and (2), becomes

$$\begin{aligned} E[\text{Turnover}(t)] &= |E[(\iota' \cdot m(t) - \iota' \cdot p(t - \Delta t)) \\ &\quad /\iota' \cdot p(t - \Delta t)]| \cdot d \cdot \Delta t \\ &= |E[(\iota' \cdot m(t - \Delta t) \cdot (1 - g \cdot \Delta t) + \iota' \cdot u(t)) \\ &\quad /\iota' \cdot p(t - \Delta t) - 1]| \cdot d \cdot \Delta t \\ &= g \cdot d \cdot \Delta t^2 \end{aligned}$$

where ι is a column vector with all elements equal to one. In the last step, we used the assumptions that $u(t)$ is independent of $p(t - \Delta t)$ and $p(t - \Delta t) = m(t - \Delta t)$. As we expected, turnover is a function of both g and d. In addition, turnover is related to the rebalancing frequency, Δt; the more frequently the portfolio is rebalanced, the lower the turnover incurred in each period. If we plug in the inferred values of $g(2.50)$, $d(3.00)$, and $\Delta t(1/12)$ in Exhibit 8 of Grinold [2007], the inferred monthly turnover is 5.21%, which is a reasonable number.

The speed of information flow is literally exogenous given the alpha model. If we also rebalance at a fixed interval, the only variable through which we can control turnover is the speed of trading, d. When the portfolio manager optimizes the turnover budget constraint, he is actually trying to select the optimal d for each market.

Multi-Market Optimization

Now, we will move to a discussion of multi-market optimization. For the sake of simplicity, assume we invest in only two markets, which will be denoted by subscripts 1 and 2. These two markets can be two regions, two countries, or two sectors. They are distinguished from each other by the alpha model we use to generate investment signals for each market.

As a global portfolio manager, our goal is to maximize the overall risk-adjusted expected returns in these two markets, net of transaction costs, as specified in the following objective function:

$$U_p = \sum_{i=1}^{2} \left\{ \alpha_{pi} - \frac{\lambda_i}{2} \cdot \omega_{pi}^2 - \frac{\chi_i}{2} \cdot \omega_{\dot{p}i}^2 \right\} \quad i = 1, 2 \qquad (5)$$

where α_{pi}, ω_{pi}, and $\omega_{\dot{p}i}$ are the expected portfolio returns, portfolio risk, and trade risk in market i, respectively. Portfolio managers might have different risk tolerance levels in different markets. For example, they can be more sensitive to the risks in developing countries than in developed ones. Therefore, we allow the risk aversion coefficient, λ_i, to vary across markets. The transaction cost level, χ_i, is also a market-specific variable. Note that in the objective function we assume the transaction costs are proportional to the trade risk.

Substituting Equations (3) and (4) into Equation (5), and recalling that $\alpha_{pi} = \omega_{pi} \cdot IR_{pi}$, we can express the objective function as

$$U_p = \sum_{i=1}^{2} \frac{d_i}{d_i + g_i} \left\{ IR_{Mi} \cdot \omega_{Mi} - \frac{\lambda_i + \chi_i \cdot g_i \cdot d_i}{2} \cdot \omega_{Mi}^2 \right\} \quad i = 1, 2 \qquad (6)$$

If we optimize over ω_{Mi} and keep IR_{Mi} and d_i fixed, the objective function becomes

$$U_p = \sum_{i=1}^{2} \frac{d_i}{d_i + g_i} \cdot \frac{IR_{Mi}^2}{2} \cdot \frac{1}{\lambda_i + \chi_i \cdot g_i \cdot d_i} \quad i = 1, 2 \qquad (7)$$

The portfolio manager's overall gain depends on the quality of the alpha model (IR_{Mi}), speed of information flow (g_i), speed of trading (d_i), transaction cost level (χ_i), and risk aversion (λ_i). Of these factors, the speed of trading

(d_i) is the only one that the portfolio manager can directly control when rebalancing a portfolio.

Returning to the turnover budget problem faced by the global portfolio manager, we assume the total turnover budget is D. Now we are trying to solve the following problem:

$$\begin{aligned} &\underset{d_1, d_2}{\text{Max}}\, U_p(d_1, d_2) \\ &st. \quad g_1 \cdot d_1 + g_2 \cdot d_2 = D \end{aligned} \tag{8}$$

Note that we impose an equality constraint on the turnover budget because in reality turnover constraints are mostly binding, especially in the multi-market case. The first-order condition yields

$$\begin{aligned} &\frac{IR_{M1}^2 \cdot g_1}{2} \cdot \frac{(\lambda_1 - \chi_1 \cdot d_1^2)}{(d_1 + g_1)^2 \cdot (\lambda_1 + \chi_1 \cdot g_1 \cdot d_1)^2} \\ &\quad = \frac{IR_{M2}^2 \cdot g_2}{2} \cdot \frac{(\lambda_2 - \chi_2 \cdot d_2^2)}{(d_2 + g_2)^2 \cdot (\lambda_2 + \chi_2 \cdot g_2 \cdot d_2)^2} \quad \text{and} \\ &g_1 \cdot d_1 + g_2 \cdot d_2 = D \end{aligned} \tag{9}$$

and basically states that, at the optimum, the marginal contribution of trading speed is equalized in each market.

What is interesting about Equation (9) is that the optimal turnover is no longer trivial, as it was in the single-market case. Grinold [2007] showed that for a single market the unconstrained optimal speed of trading is $\sqrt{\frac{\lambda}{\chi}}$, which can also be derived from Equation (9) by setting the equation to zero. In the multi-market case with a turnover budget, however, that simple solution is no longer attainable. Instead, Equation (9) defines the optimal turnover in each market as an implicit function of factors in both the local market and other markets. In addition to risk aversion and the transaction cost level, as in the single-market case, the portfolio manager should also take into account the relative quality of alpha models and the speed of information flow in the global markets; this approach intuitively makes sense.

NUMERICAL SIMULATION

To investigate how the optimal turnover level ($g_i \cdot d_i$) varies along with variables that could be considered exogenous—namely, the information ratio, speed of information flow, transaction cost level, and risk aversion in each market—we conducted a numerical experiment to test the sensitivity of optimal turnover to each of these factors.

First, we needed to decide on the setup of the experiment, or the default value of each parameter. We chose their values based on both the inferred values of Grinold [2007] and on our experience in portfolio management practice. Specifically, we assume the global turnover budget is given by $D = 10$, which roughly implies a monthly turnover rate of 7%; IR_{Mi} is set to 1, which we believe is a realistic value; and g_i is set to 2, which means about a 17% (2/12) new information flow each month. Lastly, χ_i is set to 1, and λ_i is set to 16, both of which are in line with the inferred values in Exhibit 9 of Grinold [2007].

To isolate the impact of one factor on the optimal turnover, we assume all the parameters are initially equal across markets. Then, we vary the value of the parameter being investigated in Market 1, using Market 2 as the reference point.

Information Ratio

Within the Grinold framework, the optimal turnover is independent of the information ratio implied by the alpha model. Even without the numerical experiment, however, it is easy to see from Equation (9) that d_i (and $g_i \cdot d_i$) will monotonically rise with IR_{Mi}. This relationship is also demonstrated graphically in Exhibit 1. That is, we intuitively allocate more of the turnover budget to the market where the alpha model is comparatively better at capturing information. This observation is important and makes perfect sense for global portfolio managers, but is absent in the single-market context.

Speed of Information Flow

The speed of new information flow, g_i, varies quite a bit across markets. The variation in speed can be due to the investment style selection at the country level; for example, a growth strategy in one country and a value strategy in another. It is well known that growth signals are generally more short-lived than value signals, which can result in significant differences in the pace at which alpha models update themselves, or their speeds of information flow. As expected, ceteris paribus, faster-moving information flow demands higher turnover in that market.[3]

It is worth noting, however, that instead of the linear relationship between turnover and speed of information flow demonstrated by Grinold, Exhibit 2 presents a very concave function of g_i. In other words, the optimal turnover budget should remain approximately stable even if the alpha model becomes very volatile in one market. A comparison of Exhibits 1 and 2 shows that the quality of the alpha model is more critical for the turnover budget than for the speed of information flow.

Transaction Cost Level

Global markets differ dramatically in terms of the level of transaction costs. For example, one-way trades may be executed for less than 50 basis points (bps) in large-cap U.S. names, although many trades in emerging markets cost more than 200 bps (see ITG Inc. [2008]). The final transaction costs in any market depend on two factors: 1) how expensive it is to trade (transaction

EXHIBIT 1

Sensitivity of Optimal Turnover to Information Ratio

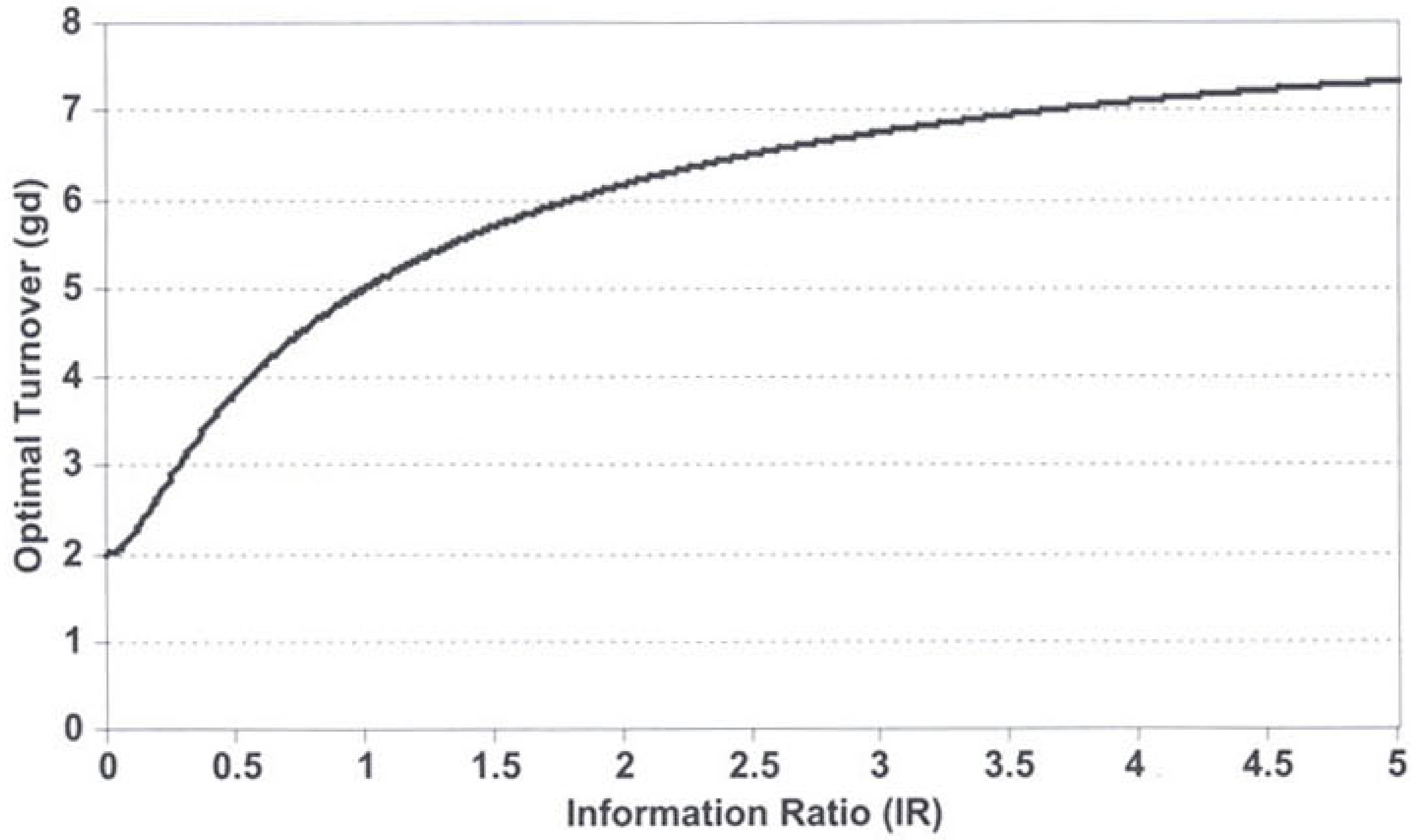
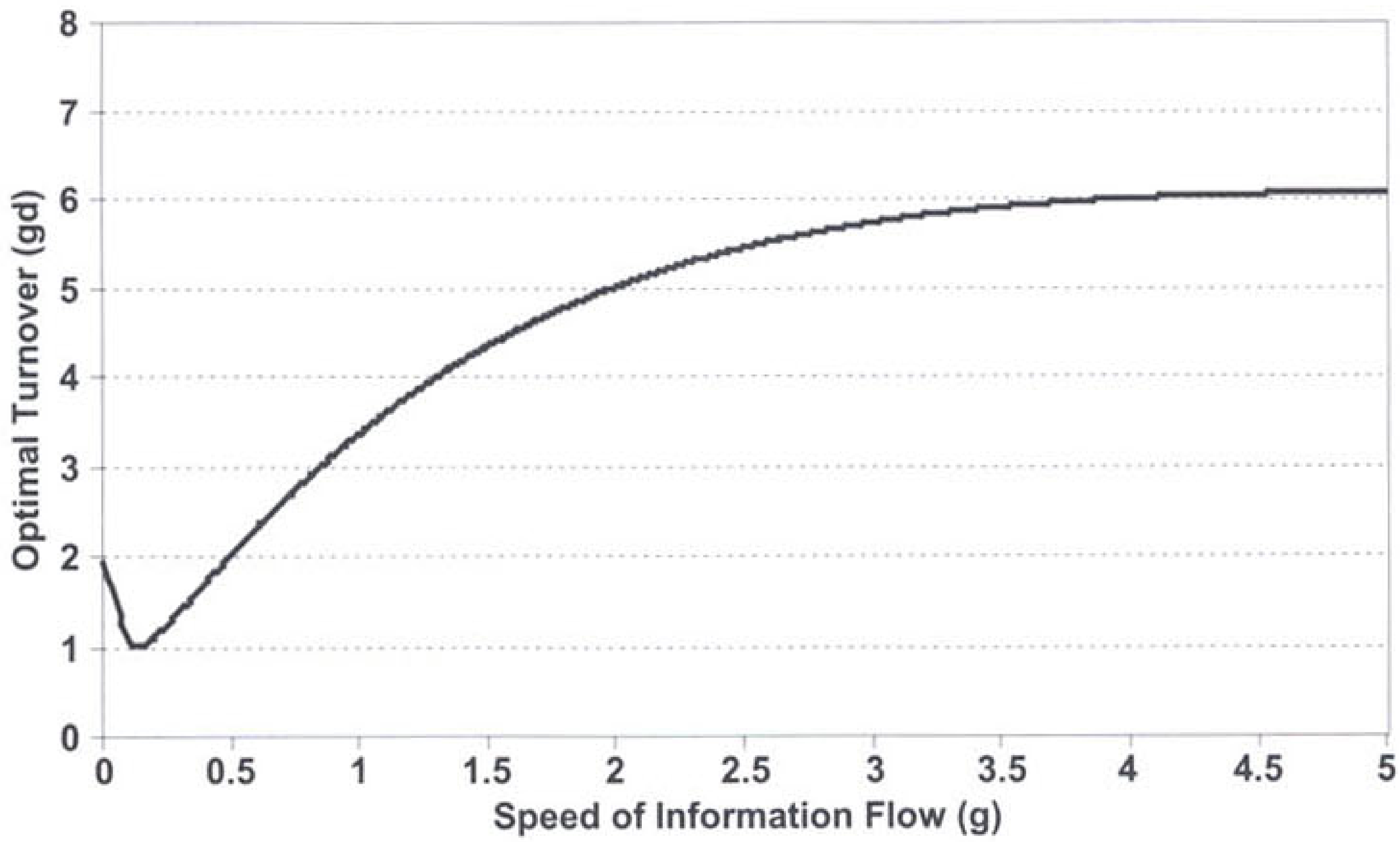


EXHIBIT 2

Sensitivity of Optimal Turnover to Speed of Information Flow



cost level) and 2) how many securities are traded (turnover). Typically, the former factor is relatively exogenous and the portfolio manager has to sacrifice turnover as the transaction cost level goes higher.[4] Exhibit 3 shows the change in the optimal turnover with the change in the transaction cost level (χ_i) in a single market. Unsurprisingly, the higher the transaction cost level, the smaller the portion of turnover budget that should be allocated to that market.

Risk Aversion

Grinold [2007] showed that optimal turnover increases with the level of risk aversion in the market, taking the form of a power function. Exhibit 4, however, tells a different story when a global turnover constraint is imposed. Specifically, the optimal turnover allocated to any market is not very sensitive to the risk aversion in that market, unless the portfolio manager's appetite for risk varies dramatically across markets, which is rarely the case.

EXHIBIT 3

Sensitivity of Optimal Turnover to Transaction Cost Level

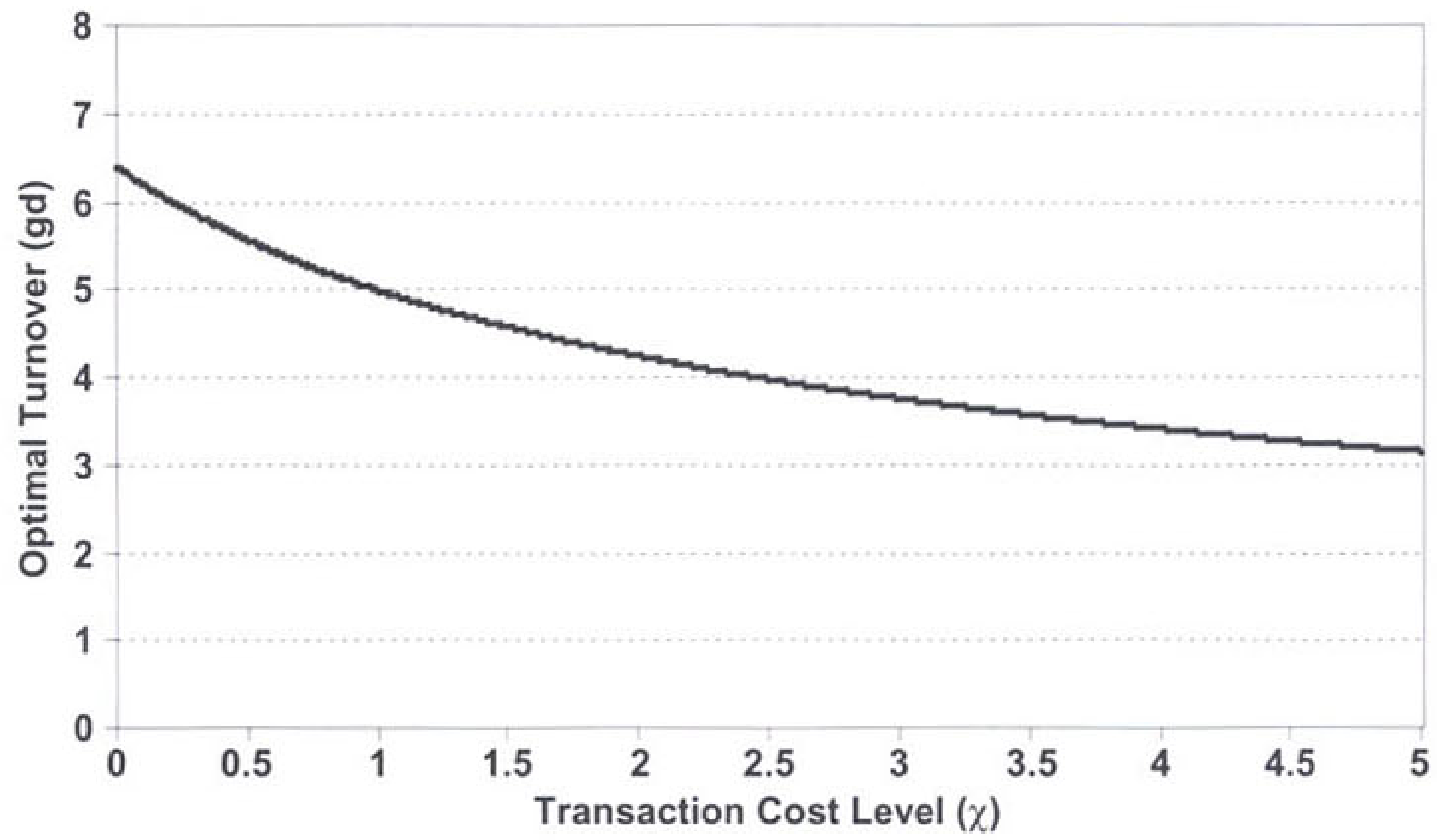
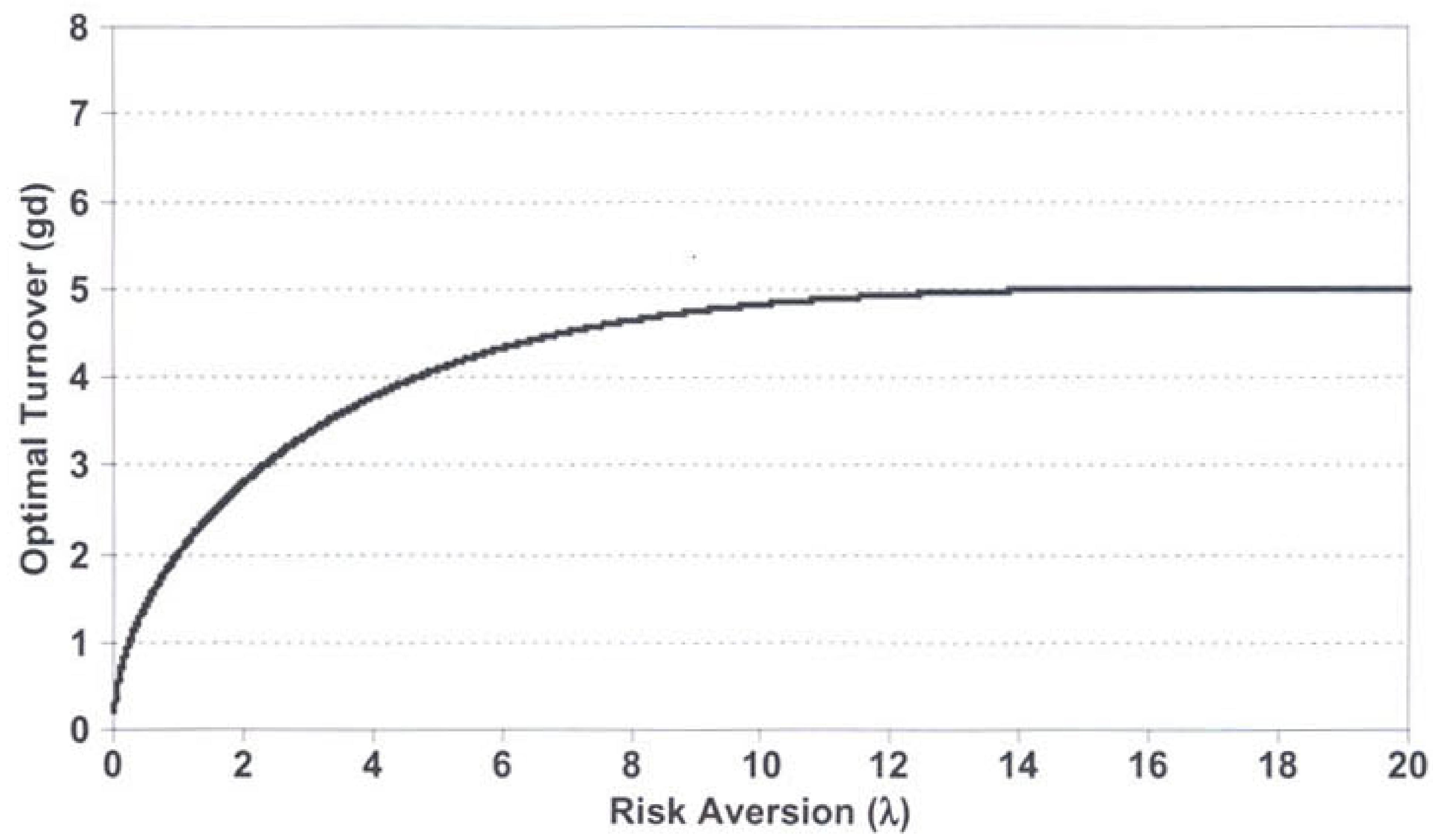


EXHIBIT 4

Sensitivity of Optimal Turnover to Risk Aversion



CONCLUSION

The efficient allocation of the turnover budget across markets is important to global portfolio managers. The existing literature, however, has not offered much theoretical guidance on this topic. In this article, we have presented a multi-market study that was conducted to provide some key insights into this topic; our study builds upon the dynamic model of Grinold [2007]. We found that the optimal turnover allocated to any market should increase with that market's quality of alpha model and speed of information flow, and decrease with the market's transaction cost level. Market-specific risk tolerance plays a marginal role according to our results.

Based on the nature of the model inherited from Grinold [2007], we took a bird's-eye view in this article. Consequently, readers should not expect the model to spit out figures that can be directly plugged into their optimizers. The findings presented in this article are generic enough, however, to serve as a compass for calibrating the turnover allocation of various investment strategies.

ENDNOTES

The views expressed in this article are those of Hao Yin, only through the period ended May 25, 2009, and do not necessarily reflect the views of State Street Corporation. All material has been obtained from sources believed to be reliable, but its accuracy is not guaranteed. The author thanks Mark Hooker, Petter Kolm, Eldar Nigmatullin, and Simon Roe for stimulating discussions and helpful comments.

[1]See Grinold [2007] Appendix A for detailed derivations of Equations (3) and (4).

[2]Strictly speaking, the expression should be $E[(\iota' \cdot |\Delta p(t)|)/|\iota' \cdot p(t-\Delta t)|]$, but we move the absolute value operation outside for the ease of exposition. Under special assumptions, Grinold [2007] derived the turnover as $\sqrt{g.d}.\Delta t$.

[3]Note that for very low speed of information flow (small value of g_i), the optimal turnover can go down as the information inflow speeds up. The reason is that for a very stable alpha model, a much wider gap (smaller d_i) is allowed between the model portfolio and actual portfolio. Because this effect overwhelms that of the increasing speed of information flow (larger g_i), the optimal turnover ($g_i d_i$) can actually go down.

[4]One trend in the industry is to incorporate transaction cost modeling in portfolio optimization. In this sense, transaction costs are becoming more and more endogenous. At the broad market level, however, the overall expensiveness to trade is still beyond portfolio managers' control.

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